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Analysis of a Novel Method for Finding Solutions of Interval Game Problems via Fuzzy Approaches

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Abstract

Game theory has significant importance in various domains as it is a powerful tool that helps the rational decision-makers to understand and assess strategic interactions. It develops mathematical models to depict these strategic engagements in competitive environments. Given the inherent uncertainty in real-world problems, obtaining exact values of payoffs for a matrix game can be difficult. In many instances, however, these payoffs vary within certain ranges, which makes interval numbers the best form to represent them. This results in the creation of a specialized category of game problem which is referred to as the interval-valued matrix game (IVMG). According to the literature, various methodologies exist to identify the optimal strategies and game value for IVMGs. However, many of these methods possess some shortcomings discussed in the paper, highlighting the requirement for a novel approach. Therefore, in the present study, we propose a novel method to find solutions to game problems with interval payoffs, utilizing the fuzzy concept. Since operations and comparisons on interval numbers are not well-defined, we transform the elements of payoff matrix into a fuzzy representation. Utilizing ranking function for defuzzification of these fuzzy payoffs, we transform them to crisp form. The solution for the subsequent crisp matrix game is obtained using a graphical technique or linear programming problem approach. Additionally, numerical examples are provided for validation of the presented method. The game values for these examples are also obtained using methods presented by other researchers in the literature, and comparisons with these methods are made, highlighting the limitations of the methods in the existing literature and significance of the presented method. Finally, conclusions with shortcomings and future scope of research based on the paper are described.

Keywords: Generalized pentagonal fuzzy number, Generalized hexagonal fuzzy number, Fuzzification, Ranking function, Two person zero-sum game, Value of game.

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1|Introduction

Decision makers often face the situation, competitive in nature and with conflicting interests of the parties/competitors, such that the decisions made by one party is dependent on the decisions made by the other party. Optimization of these type of situations situations which are competitive and conflicting in nature is done using theory of games, whose formulation in mathematical terms was done initially by Morgenstern and Neumann [49]. The theory of games is a modeling technique which is considerably significant in numerous disciplines as it serves as a rich tool for understanding and investigating strategic engagements between rational decision-makers.

In the field of computer science, it is crucial for creating algorithms and methods for artificial intelligence. In realms of economics, it assists in designing models for competitive markets and determining the best possible results. In ecology and biology, it sheds light on the survival strategies of various species and provides researchers a means to analyze various animal behaviors. The broad spectrum of its applications makes game theory a crucial analytical tool, deepening our insight into strategic decision-making (DM) throughout diverse real-world scenarios. There are different categories of mathematical games that have been thoroughly researched and effectively utilized in a broad spectrum of areas.

A prime example is the two-person zero-sum finite game, in which the opponents have a limited set of strategies, and the game's rule is that the profit any player gains is precisely matched by the other player's loss. In the most straightforward scenario, the entries in the payoff matrix (PM) are precise values. However, accurately determining these payoffs is difficult due to the uncertain information and the players' incomplete understanding of the situation, causing them to estimate the payoff values with some level of imprecision This is managed by considering payoffs in various forms according to the available information.

To tackle this issue, various types of numbers are used to represent payoffs, including fuzzy number (FN) [51] and their extensions like neutrosophic numbers [47], intuitionistic FNs [2], Pythagorean FNs [50], hesitant FNs [52], Fermatean FNs [45], and picture FNs [11]. This leads to the formulation of game problem (GP) in diverse uncertain environments. An example of this formulation is the IVMG [35]. In the realm of IVMG, interval values are substituted in place of crisp values in the PM of the classical game, to accommodate the inherent uncertainty of data from real-world scenarios. This representation provides a more practical depiction of ambiguity by accounting for the lower as well as the upper bounds of possible values. Providing a wider view of the potential outcomes in strategic interactions, IVMG permits decision-makers to manage complex scenarios in which exact consequences are difficult to ascertain.

A lot of researchers have explored IVMG problems in their research work as well as presented various approaches to identify the best strategies and the game's value. Fuzzy approach is one of the methodologies that is used to address a range of DM problems where uncertainty is represented by intervals. A graphical method for IVMG was presented by Nayak and Pal [37]. In 2008, Collins and Hu [10] considered game matrix with interval values and extended the findings obtained from classical matrix game (MG) to IVMG which are fuzzily determined. Nayak and Pal [38] solved GPs with payoffs as interval numbers by formulating an interval linear programming model (LPM) and developed an algorithm based on multi-section method.

Li [28] also considered the same type of problem and based on the upper as well as the lower bounds of the payoff entries, formulated a pair of LPMs, which, when solved, yields the upper and lower bounds of the game's value. Also, Li et al. [31] defined interval inequality relations and generated auxiliary interval programming models for being utilized in the considered IVMG. They then used the established fuzzy ranking index to solve the resulting pair of bi-objective LPMs and identify the optimal solution of the game. A multi-objective linear programming approach that makes use of the upper as well as the lower bounds of payoffs to address an IVMG was presented by Roy and Mondal [39]. A bi-linear programming approach was introduced by Fei and Li [15] to address interval bi-MGs, aiming to assess

the upper and lower limits of optimal strategies for participants along with the game's value. In 2020, Dey and Zaman [14] presented a novel optimization technique to address a two-person zero-sum as well as a nonzero-sum game with incomplete information, in which payoffs are available as multiple intervals. Interval-valued games can emerge in various real-world game-theoretic scenarios, such as marketing strategy games [31, 16], investment DM [10, 28], tourism management [6], sensor selection problems [13] and joint replenishment applications [3].

Decision-makers often provide linguistic descriptions of the parameters. Fuzzy set theory [51] is crucial for representing imprecision, particularly the ambiguity associated with natural language.. As a result, FNs and operators on them were developed, which form a mathematical foundation of application of FST. Fuzzy numbers have been widely employed in various DM problems such as hospital site selection problem [17], mitigating labor shortages in post-pandemic logistics [36], transportation problems [8, 7], Ph.D. supervisor selection problems [33], for ranking hotel locations [18], for optimal site-selection problem [1], for pre-diagnosis of disease based on symptoms [34]. In this research, we have also presented a technique for GP containing interval payoffs (IPs) using fuzzy approach. A novel fuzzification approach is presented in the paper to transform the interval problem to fuzzy problem. Many other authors have used fuzzy approach to find a solution to decision making problems in the past. To find the optimal solutions for two player zero-sum MGs with triangular fuzzy payoffs (FP), Li [26] developed two auxiliary LPPs by incorporating operations defined on triangular fuzzy numbers (TFN). They further utilized the order relation of these numbers to formulate three-objective LPMs and determined the minimax and maximin strategies for the players using the lexicographic method, while the value of the MG was calculated using the simplex method. Li and Hong [30] presented a method to solve MGs with constrained strategies and payoffs as TFNs. Li [27] created a method for MG with triangular FPs which always guarantees a common value for the players' loss-ceiling and gain-floor, which is also the fuzzy value of game. They also obtained the mean, as well as the upper and lower limits of TFN by computing the solution of LP models developed using data from both the 0-cut set as well as the 1-cut set of FPs. Furthermore, for any alpha-cut set of the TFN-type fuzzy value, they determined the lower bounds, upper bounds as well as the optimal strategies for the participants, by finding a solution to the developed LP models at a mentioned confidence level alpha. Chandra and Aggarwal [9] identified a limitation in Li's [27] method of transforming fuzzy GP into deterministic linear programming problem (LPP). They offered a correction and introduced a novel technique to solve two-person zero-sum MGs that have piecewise linear FPs. Additionally, they presented important concepts for addressing MGs with triangular FPs and developed auxiliary LPMs and methods to solve these games. Using interval parametric technique, Sahoo [40] presented a method for fuzzy MGs. Sahoo [41], Selvakumari and Lavanya [44] presented methods to solve MGs with trapezoidal or triangular FPs by transforming problems to crisp MGs using defuzzification methods. Li [29] introduced new concepts and approaches for solving MGs with triangular FPs by developing auxiliary LPMs for these games. In the same year, Kumar et al. [21] presented a method for trapezoidal fuzzy MGs, where the game is first converted into a fuzzy LPP and then further transformed into three optimization problems. The solution of these problems give the optimal strategies and optimal game value. By applying "measure" ranking method, Thirucheran and Lavanya [48] presented a method to solve fuzzy MG. Jana and Kumar [20] studied MGs with generalized trapezoidal FPs and presented a solution method utilizing a ranking function. Bigdeli et al. [4] used Karush-Kuhn-Tucker conditions and idea of nearest interval approximation of FNs to solve GP. Khalifa [22] used linear programming approach for two-person zero-sum MG with FPs. He transformed FPs to deterministic payoffs using Rouben's method and then formulated a pair of LPPs to obtain optimal strategies for both participants. Bisht and Dangwal [8] presented a method to find optimal solution of transportation problems with parameters in the form of intervals by transforming the problem into octagonal fuzzy form. Seikh et al. [43] considered a MG with dense fuzzy lock payoffs. They obtained the optimal solution of such games by formulating a pair of auxiliary dense fuzzy programming problems and further transforming them into two equivalent crisp LPPs, which are solved using LINGO. Seikh and Dutta [42] presented a study on MGs with single-valued trapezoidal neutrosophic payoffs. To identify the optimal mixed strategies and the game's value, they developed two types of neutrosophic mathematical programming problems, that was then transformed into interval-valued multi-objective programming problems with the help of a ranking function, resolved them by applying the weighted average approach and the solution was obtained using LINGO.

In addition to game theory, FNs have recently been utilized to handle the uncertainty in various other DM problems. Recently, Das [12] considered a linear fractional programming problem where the objective function and right-hand side values are symmetric trapezoidal fuzzy numbers (TrapFN), left-hand side constraints are real numbers and the decision variables are non-negative TrapFN. They utilized the ranking function along with

the operations of TrapFN to obtain a crisp problem from the fuzzy problem and further used Swarup to solve the program. Later, Khalifa [23] presented a solution method to determine the minimization of the expected makespan for a two-machine flow shop scheduling problem with piecewise quadratic fuzzy processing time. Jain et al. [19] combined transportation scheduling and inventory control to enhance the efficiency of the entire supply chain. They presented the idea of the overtime fuzzy inventory-transportation problem, emphasizing the development of an optimal distribution strategy that that takes overtime into account to minimize overall distribution costs. To handle the uncertainties and ambiguities inherent in real-world situations, they employed TrapFNs. Bisht et al. [5] presented a novel method to find an optimal solution of generalized triangular intuitionistic fuzzy transportation problem. In addition, Khalifa [25] presented an interactive procedure for obtaining optimal compromised solution for a problem of vendor selection having fuzzy parameters, viz., upper limit of the quantity available, price of a unit item and an aggregate demand for the item. Khalifa et al. [24] presented an approach on multi-objective De Novo programming problem where the objective function has piecewise quadratic fuzzy number (PQFN) coefficients. They applied close interval approximation of the PQFN along with a min-max goal programming approach, incorporating both positive and negative ideals, to achieve an optimal compromise system design.

1.1|Novelty of the presented study

In the present study, we identify the optimal strategies obtained by solving two-person zero-sum interval non-cooperative GPs using a fuzzy approach. As mentioned above, many authors in the literature have presented different solution approaches for IVMG, however none have utilized fuzzy approach for the MG, where the interval data is transformed into FN by applying some fuzzification method. This is done since most of the existing methods for comparing and ranking intervals are imprecise and lack effectiveness, particularly when dealing with nested and fully or partially overlapping intervals. Also, it has been pointed out by few authors that some of the ranking approaches for intervals do not coincide with the requirements of rational decision-makers. Thus, for overcoming these challenges, IPs are transformed to FNs using relevant fuzzification methods. The pentagonal fuzzy approach is appropriate to use in conditions where the decision maker gives linguistic description of the payoffs in such a way that the uncertainty or imprecision is not only expressed as a range between two limits but there is also some vague information about five distinct points in that interval. Similarly, in case of six data points, hexagonal fuzzy approach can be used. This approach provides a balance between simplicity and precision, making it suitable for DM scenarios where the confidence levels at five/six specific points need to be considered. Thus, we introduce a novel fuzzification method that transforms the interval entries of the PM into a fuzzy format. The fuzzy entries are then transformed to crisp values, effectively transforming the IVMG into a classical GP. Recognizing the similarity between linear programming and GPs, we start by generating LPPs associated to the game, which we then solve using simplex method or we use the traditional graphical method to identify the best strategies. As a result, we determine the game's value in both interval and crisp forms, along with the players' optimal strategies in crisp form. For a two-person zero-sum MGs in which the payoffs are expressed in generalized pentagonal or hexagonal fuzzy forms, if we want to determine the optimal strategies, then the presented method is very suitable. This representation enables data analysts to convey information with two levels of confidence. Additionally, generalized pentagonal or heptagonal FNs offer greater flexibility in managing data, bearing in mind the opinions of both the decision maker and the data analyst regarding the degree of confidence.

The primary findings of this research work are:

1. For fuzzification of intervals, a novel method is presented that can also be utilized in other DM problems.

2. A novel method is introduced and utilized in solving interval-valued game problems (IVGP).

3. This presented method is applicable for determining the optimal strategies and the game value for pentagonal and hexagonal fuzzy GPs as well.

4. A real-world example of an IVGP is demonstrated within the context of a competitive retail environment.

5. A review of the existing algorithms for IVMG is carried out, with a detailed examination of their limitations.

The remaining paper is arranged in the subsequent way: In Section 2 Basic definitions of some concepts such as fuzzy sets, FNs, interval and crisp GP, saddle point are given. In Section 3, fuzzification method for transforming interval values to pentagonal fuzzy form and hexagonal fuzzy form is defined. Also, a ranking function for conversion of the data in these two fuzzy forms to crisp form is defined in this section. Further, in Section 4, a solution procedure for solving interval GP and obtaining the optimal mixed strategies along with the game's value is defined. In Section 5, two numerical examples of IVMG are provided to demonstrate this procedure. Then the computational results for these examples are evaluated using some existing methods and are discussed in Section 6. Conclusions along with the future research scope and some limitations are given in Section 7.

2|Preliminaries

Definition 1 (Crisp Game Problem [49]). *Suppose G is a game involving two players, Player A and Player B, where the pure strategies for Player A and Player B are represented by P_1 and P_2, in the same order. Then*

$$P_1 \ = \ \{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_r\}, \ P_2 \ = \ \{\beta_1, \beta_2, \beta_3, ..., \beta_s\}.$$

Consider for Player A and Player B, the sets S and T, in the same order, whose elements are the mixed strategies. Moreover, suppose q_j; $j = 1, 2, 3, ..., s$ and p_i; $i = 1, 2, 3, ..., r$ represent the chances of selecting $q_j \in P_2$; $j = 1, 2, 3, ..., s$ by Player B and $\alpha_i \in P_1$; $i = 1, 2, 3, ..., r$ by Player A, in the same order. Then

$$S \ = \ \{p \ = \ (p_1, p_2, p_3, ..., p_r)^T \ \in \mathcal{R}^r : \ \sum_{i=1}^{r} p_i \ = \ 1, \ p_i \geq 0, \ i = 1, 2, 3, ..., r\},$$

$$T \ = \ \{q \ = \ (q_1, q_2, q_3, ..., q_s)^T \ \in \mathcal{R}^s : \ \sum_{j=1}^{s} q_j \ = \ 1, \ q_j \geq 0, \ j = 1, 2, 3, ..., s\},$$

where the transpose of x is denoted by x^T, and the s-dimensional and r-dimensional Euclidean spaces are denoted by R^s and R^r, in the same order.

Additionally, if the PM of Player A is represented by A, then

$$A \ = \ \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1s} \\ a_{21} & a_{22} & \dots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rs} \end{bmatrix} \tag{1}$$

The two-person zero-sum game is represented by $G \equiv (S, T; A, \leq)$.

Definition 2 (Interval Number [10]). *Any subset of real numbers, which is closed, denoted by $\mathcal{A} = [\underline{a}, \ \overline{a}] = \{x \in \mathbb{R} : \underline{a} \leq x \leq \overline{a}\}$, where $\underline{a} \leq \overline{a}$, and $\underline{a}$ and $\overline{a}$ are the lower and upper bounds of interval $\mathcal{A}$, in the same order, is called an interval.*

Definition 3 (Interval Matrix Game [10]). *Let Player A be an optimistic player and Player B be a pessimistic player such that Player B tries to minimize the loss and Player A tries to make maximum profit. Then a two person, non-zero sum interval MG is described by a matrix of order $m \times n$ whose entries are in the form of intervals as defined below:*

$$\mathcal{A} \ = \ \begin{bmatrix} [\underline{a}_{11}, \overline{a}_{11}] & [\underline{a}_{12}, \overline{a}_{11}] & \dots & [\underline{a}_{1n}, \overline{a}_{1n}] \\ [\underline{a}_{21}, \overline{a}_{21}] & [\underline{a}_{22}, \overline{a}_{22}] & \dots & [\underline{a}_{2n}, \overline{a}_{2n}] \\ \vdots & \vdots & \ddots & \vdots \\ [\underline{a}_{m1}, \overline{a}_{m1}] & [\underline{a}_{m2}, \overline{a}_{m2}] & \dots & [\underline{a}_{mn}, \overline{a}_{mn}] \end{bmatrix}. \tag{2}$$

Also, for Player A and Player B, let X_m, Y_n be set of strategies, in the same order. Let $x \in X_m$ and $y \in Y_n$ be strategies of Player A and Player B, then expected gain of Player A is

$$G(x, y) = x^T A y = \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} x_i \left[\underline{a}_{ij}, \overline{a}_{ij}\right] y_j.$$

Now, let $\overline{\nu}$ *and* $\underline{\nu}$ *be the upper and lower value of the game, then.*

$$\underline{\nu} = \underset{x}{Min}\ \underset{y}{Max}\ G(x,y).$$

$$\overline{\nu} = \underset{y}{Max}\ \underset{x}{Min}\ G(x,y).$$

Definition 4 (Saddle Point [49])**.** *It is claimed that a game contains a saddle point if in that game minimax for Player B and maximin for Player A are equal. If* (r,s)th *position of the PM* $\mathcal{A}_{\times} = [[\underline{a}_{ij}, \overline{a}_{ij}]]_{\times}$ *is a saddle point, then*

$$[\underline{a}_{rs}, \overline{a}_{rs}] = \min_{j}\left\{\max_{i} [\underline{a}_{ij}, \overline{a}_{ij}]\right\} = \max_{i}\left\{\min_{j} [\underline{a}_{ij}, \overline{a}_{ij}]\right\}$$

Definition 5 (Fuzzy Set [51])**.** *Let* $\mu_{\tilde{\Gamma}}(x) : \tilde{\Gamma} \rightarrow [0,1]$ *be the membership function (MF) on a set* $\tilde{\Gamma}$*. Then the pair* $(\tilde{\Gamma}, \mu_{\tilde{\Gamma}})$ *is called a fuzzy set. A fuzzy set* $\tilde{\Gamma}$ *is often described as* $\tilde{\Gamma} = \{(x, \mu_{\tilde{\Gamma}}(x)) : x \in \Gamma,\ \mu_{\tilde{\Gamma}}(x) \in [0,1]\}$*.*

Definition 6 (FN: Fuzzy Number [51])**.** *Let* $\mu_{\Gamma^*} : \mathbb{R} \rightarrow [0,1]$ *be the MF of a fuzzy subset* Γ^* *within* $\mathbb{R}$*. Then* Γ^* *is an FN if*

(a) $\mu_{\Gamma^*}(t)$ *is piecewise continuous in domain.*

(b) $\exists$ *some* $t_0 \in \mathbb{R}$ *such that* $\mu_{\Gamma^*}(t_0) = 1$*, i.e.,* Γ^* *is normal.*

(c) $\mu_{\Gamma^*}(\alpha t_1 + (1-\alpha)t_2) \geq min\ (\mu_{\Gamma^*}(t_1), \mu_{\Gamma^*}(t_2))\ \forall\ t_1, t_2$ *in* $\mathcal{R}$*,* $\alpha \in$ *[0,1], i.e.,* Γ^* *is convex.*

Definition 7 (GPFN: Generalized Pentagonal Fuzzy Number [7])**.** *An FN* $\tilde{A}$ *is known as a GPFN (see Fig. 1) represented by* $\tilde{A} = (p_1, p_2, p_3, p_4, p_5; u, v)$*, where* $p_1,\ p_2,\ p_3,\ p_4,\ p_5,\ u,\ v \in \mathbb{R}$ *if* $p_1 < p_2 < p_3 < p_4 < p_5$*,* $0 < u < v \leq 1$ *and its MF* $\mu_{\tilde{A}}(x)$ *is such that*

$$\mu_{\tilde{A}}(x) = \begin{cases} u\left(\dfrac{x-p_1}{p_2-p_1}\right), & if\ \ p_1 \leq x \leq p_2; \\ u + (v-u)\left(\dfrac{x-p_2}{p_3-p_2}\right), & if\ p_2 \leq x \leq p_3; \\ v, & if\ x = p_3; \\ v + (u-v)\left(\dfrac{x-p_3}{p_4-p_3}\right), & if\ p_3 \leq x \leq p_4; \\ u\left(\dfrac{x-p_5}{p_4-p_5}\right), & if\ p_4 \leq x \leq p_5; \\ 0, & otherwise. \end{cases}$$

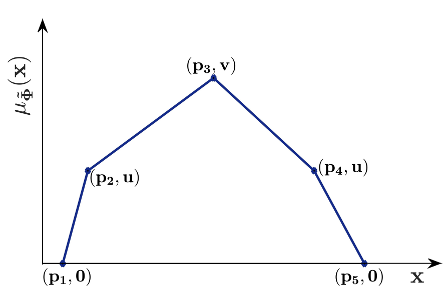

Fig. 1. Representation of GPFN.

Definition 8 (GHFN: Generalized Hexagonal Fuzzy Number [7])**.** *An FN* $\tilde{A}$ *is known as a GHFN (see Fig. 2) represented by* $\tilde{A} = (h_1, h_2, h_3, h_4, h_5, h_6; u, v)$*, where* $h_1,\ h_2,\ h_3,\ h_4,\ h_5,\ h_6,\ u,\ v \in \mathcal{R}$ *if*

$h_1 < h_2 < h_3 < h_4 < h_5 < h_6$, $0 < u < v \leq 1$ *and its MF* $\mu_{\tilde{\mathsf{h}}}(x)$ *is given as:*

$$\mu_{\tilde{\mathsf{A}}}(x) = \begin{cases} u\left(\dfrac{x-h_1}{h_2-h_1}\right), & \text{if } h_1 \leq x \leq h_2; \\ u+(v-u)\left(\dfrac{x-h_2}{h_3-h_2}\right), & \text{if } h_2 \leq x \leq h_3; \\ v, & \text{if } h_3 \leq x \leq h_4; \\ v+(u-v)\left(\dfrac{x-h_4}{h_5-h_4}\right), & \text{if } h_4 \leq x \leq h_5; \\ u\left(\dfrac{x-h_6}{h_5-h_6}\right), & \text{if } h_5 \leq x \leq h_6; \\ 0, & \text{otherwise.} \end{cases}$$

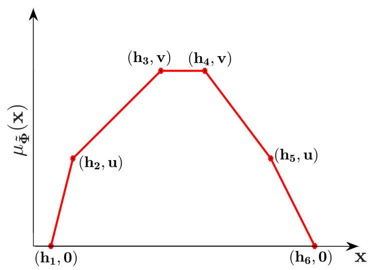

Fig. 2. Representation of GHFN.

3|Fuzzification and Ranking Function

3.1|Fuzzification method

In some game theory problems, the payoffs corresponding to different strategies needs to be considered as intervals when dealing with variability, uncertainty or imprecision in data. This situation arises due to various reasons such as incomplete information when payoffs are to be decided much ahead of time, uncertain environmental factors like weather and resource availability, and fluctuating market conditions which impacts prices as well as demand of products. Additionally, players also declare IPs since financial models are affected by risk and uncertain future events, and there might be variable offers due to negotiations and bidding processes. Also, sometimes, data collection methods yield approximate values, which also turns out to be a possible cause for IPs. Consequently, interval-valued data is required by players for communicating their payoffs beforehand to select successful strategies as it encompasses a range between two values. Given the difficulty of specifying precise payoff values beforehand, using intervals to represent them is frequently a more feasible approach. Dealing with interval-valued data during DM imposes difficulties as the operations that are defined over the collection of intervals are not well-defined. This lack of clarity makes tasks such as ranking and comparison challenging and impractical. As a result, traditional techniques used for solving precise two-person zero-sum games are not easily applicable.

Fuzzification of intervals involves transforming interval-valued data into FNs, which are more effective for handling imprecision and uncertainty in DM problems. Fuzzification is applied because FNs offer a robust method for IVGPs by providing well-defined ranking and comparison. They facilitate better mathematical manipulation due to more precise arithmetic operations as compared to intervals. Also, fuzzification captures the inherent uncertainty and imprecision in the data more effectively, reflecting real-world conditions. Furthermore, fuzzification aids improved decision making by providing a clearer picture of the range and likelihood of different outcomes and with the help of fuzzy approach we are also allowed to apply any conventional method from the literature for obtaining the optimal solutions.

Its mechanism starts with identification of lower and upper bounds of the intervals which represent the range of possible values. For fuzzification to pentagonal fuzzy number (PFN), we select three intermediate points within the interval, say $c,\ d,\ e$, to create the pentagonal shape, while for fuzzification to hexagonal fuzzy numbers (HFN), we select four intermediate points within the interval, say $c,\ d,\ e,\ f$, to create the hexagonal shape. These points are chosen based on the distribution of data or specific criteria relevant to the problem. In case of PFN, The points c and e represent lower and upper significant values of data within the interval with membership degree $\frac{1}{2}$, while d represents the value within the interval for which membership degree is maximum, i.e., 1. For HFN, points c and f represent some significant percentiles within the interval with membership degree $\frac{1}{2}$, while d and e represent the data points within the interval between which the membership degree is maximum, i.e., 1. Lastly, we establish the MF for PFN and HFN, indicating the amount to which each value, that is lying within the interval, belongs to the fuzzy set. This method provides a comprehensive way to handle uncertainty, enabling more detailed and reliable DM processes. Mathematically, this mechanism is described as:

Let $A = [a, b]$ be an interval number. This number can be transformed into pentagonal and hexagonal numbers as follows:

(i) PFN: To assign values to three intermediate values mentioned above, consider $\Gamma = \dfrac{b-a}{7}$. Then PFN corresponding to interval A is

$$A \equiv \tilde{A} = (a,\ c,\ d,\ e,\ ,b;\ \tfrac{1}{2},\ 1) = (a,\ a+\Gamma,\ a+3\Gamma,\ a+5\Gamma,\ b;\ \tfrac{1}{2},\ 1).$$

We define the MF for PFN $(a,\ c,\ d,\ e,\ ,b;\ \frac{1}{2},\ 1)$ as:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{1}{2}\left(\dfrac{x-a}{c-a}\right), & \text{if } \ a \le x \le c; \\ \dfrac{1}{2}\left(1+\dfrac{x-c}{d-c}\right), & \text{if } c \le x \le d; \\ 1, & \text{if } x = d; \\ 1-\dfrac{1}{2}\left(\dfrac{x-d}{e-d}\right), & \text{if } d \le x \le e; \\ \dfrac{1}{2}\left(\dfrac{x-b}{e-b}\right), & \text{if } e \le x \le b; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) HFN: To assign values to four intermediate values mentioned above, consider $\Gamma = \dfrac{b-a}{9}$. Then HFN corresponding to interval A is

$$A \equiv \tilde{A} = (a,\ c,\ d,\ e,\ ,f\ ,b;\ \frac{1}{2},\ 1) = (a,\ a+\Gamma,\ a+3\Gamma,\ a+5\Gamma,\ a+7\Gamma,\ b;\ \frac{1}{2},\ 1).$$

We define the MF for HFN $(a,\ c,\ d,\ e,\ ,f\ ,b;\ \dfrac{1}{2},\ 1)$ as:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{1}{2}\left(\dfrac{x-a}{c-a}\right), & \text{if } \ a \le x \le c; \\ \dfrac{1}{2}\left(1+\dfrac{x-c}{d-c}\right), & \text{if } c \le x \le d; \\ 1, & \text{if } d \le x \le e; \\ 1-\dfrac{1}{2}\left(\dfrac{x-e}{f-e}\right), & \text{if } e \le x \le f; \\ \dfrac{1}{2}\left(\dfrac{x-b}{f-b}\right), & \text{if } f \le x \le b; \\ 0, & \text{otherwise .} \end{cases}$$

3.2|Ranking Function for Fuzzy Numbers [7]

Ranking functions for FNs are essential tools that facilitate accurate quantification, comparison, and DM in the presence of uncertainty, thereby playing a crucial role in various DM problems. When the data of a problem is considered as FNs, it often becomes necessary to compare and quantify them while making decisions. Using a suitable ranking function is important for achieving correct results, as inappropriate rankings can often result in inaccurate conclusions. Therefore, ranking is an important part of the DM process, motivating many researchers to develop various novel approaches for ranking of FNs. However, comparing and ordering FNs can be challenging. To address this, the natural order of real numbers is extended to FNs by transforming them into real numbers. Define the function $R : \tilde{A}(\mathbb{R}) \to \mathbb{R}$, where $\tilde{A}(\mathbb{R})$ is a set of FNs which is defined on the real line $\mathbb{R}$, that maps FNs to unique real numbers. This R is known as the ranking function.

3.2.1|Ranking Function For Generalized Pentagonal Fuzzy Numbers

Consider a GPFN $\tilde{\Phi} = (p_1,\ p_2,\ p_3,\ p_4,\ p_5;\ v/2,\ v)$ as shown in Fig. 1 considering $u = \dfrac{v}{2}$. Let the intersection point of extended PQ and extended SR be A(i,j). Then,

$$i = \frac{p_1p_4 - 2p_2p_4 + p_2p_3}{p_1 - p_2 + p_3 - p_4}, \qquad j = \frac{v}{2}\left(\frac{i - p_1}{p_2 - p_1}\right). \tag{3}$$

Let the intersection point of extended QR and extended TS be B(k,l). Then,

$$k = \frac{p_2p_5 - 2p_2p_4 + p_3p_4}{p_3 - p_2 + p_5 - p_4}, l = \frac{v}{2}\left(\frac{k - p_5}{p_4 - p_5}\right). \tag{4}$$

Further, join S to P and T to Q & let C(e,f) and D(g,h) be centroids of $\triangle PAS$ and $\triangle QBT$, in the same order (see Fig. 3 & Fig. 4). Then

$$e = \frac{p_1 + p_4 + i}{3}, f = \frac{j + \frac{v}{2}}{3}, \tag{5}$$

$$g = \frac{p_2 + p_5 + k}{3}, h = \frac{l + \frac{v}{2}}{3}. \tag{6}$$

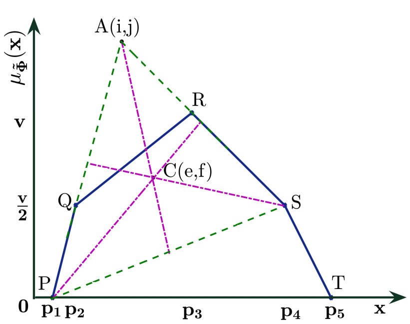

Fig. 3. Midpoint of $\triangle$ PAS.

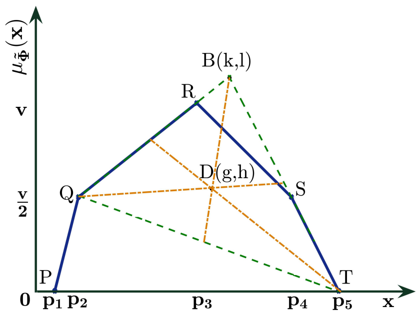

Fig. 4. Midpoint of △ QBT.

Note 1. *On comparing the GPFN obtained by the fuzzification method and the form considered in the ranking function, it is clear that $\nu = 1$. Thus, the value of ν should be considered in expressions* (3), (4), (5) *and* (6).

Now, let the mid point of $\overline{CD}$ be denoted by $M(\alpha, \beta)$. Then,

$$\alpha = \frac{e+g}{2}, \ \beta = \frac{f+h}{2}.$$

So, ranking value of GPFN $\tilde{\Phi}$ is given by (Fig. 5):

$$\mathbf{R}(\tilde{\mathbf{\Phi}}) = \sqrt{\alpha^{\mathbf{2}} + \beta^{\mathbf{2}}}. \tag{7}$$

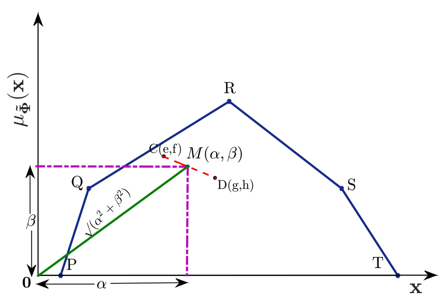

Fig. 5. Ranking of GPFN.

Note 2. *The presented ranking function is derived from the distance of mid point M of centroids C and D from the origin O.*

3.2.2|Ranking For Generalized Hexagonal Fuzzy Numbers

Consider a GHFN as shown in Fig. 2 and let the point of intersection of extended PQ and extended TS be A(i,j). Then,

$$i = \frac{h_1 h_5 - 2h_2 h_5 + h_2 h_4}{h_1 - h_2 + h_4 - h_5}, j = \frac{v}{2}\left(\frac{i - h_1}{h_2 - h_1}\right). \tag{8}$$

And, let the intersection point of extended QR and extended UT be B(k,l). Then,

$$k = \frac{h_2 h_6 - 2h_2 h_5 + h_3 h_5}{h_3 - h_2 + h_6 - h_5}, l = \frac{v}{2}\left(\frac{k - h_6}{h_5 - h_6}\right). \tag{9}$$

Further, suppose D(r,s) and C(p,q) be centroids of $\triangle QBU$ and $\triangle PAT$, in the same order, as shown in Fig. 6 and Fig. 7. Then,

$$p = \frac{h_1 + h_5 + i}{3}, q = \frac{j + \frac{v}{2}}{3}. \tag{10}$$

$$r = \frac{h_2 + h_6 + k}{3}, s = \frac{l + \frac{v}{2}}{3}. \tag{11}$$

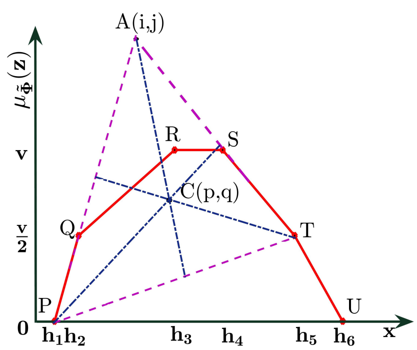

Fig. 6. Midpoint of △PAT

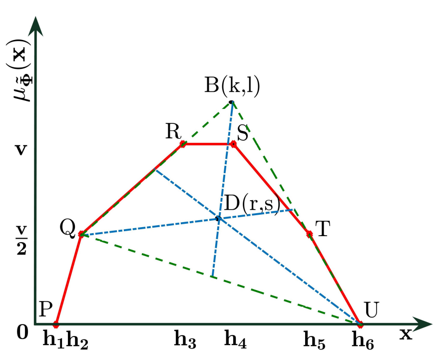

Fig. 7. Midpoint of △ QBU

Note 3. *On comparing the GPFN obtained by the fuzzification method and the form considered in the ranking function, it is clear that* $\nu = 1$. *Thus, the value of* ν *should be considered in expressions* (8), (9), (10) *and* (11).

Now, let the mid point of $\overline{CD}$ be denoted by $M(\alpha, \beta)$. Then,

$$\alpha = \frac{p + r}{2}, \beta = \frac{q + s}{2}.$$

Thus, ranking value of GHFN $\tilde{\Phi}$ is given by (Fig. 8):

$$\mathbf{R}(\tilde{\boldsymbol{\Phi}}) = \sqrt{\alpha^{\mathbf{2}} + \beta^{\mathbf{2}}}. \tag{12}$$

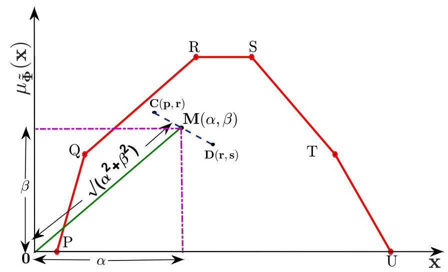

Fig. 8. Ranking of GHFN

Note 4. *The presented ranking function is derived from the distance of mid point M of centroids C and D from the origin O.*

4|Procedure of solution

In this section, a novel technique to solve an IVMG is provided, along with a corresponding flowchart, as outlined below:

Step 1: Using the presented fuzzification method, transform the given IPs of the GP to FPs, i.e., if $\tilde{A_{ij}} = (a^1_{ij}, a^2_{ij}, a^3_{ij}, a^4_{ij}, a^5_{ij};\ \frac{1}{2},\ 1)$ denotes the PFN corresponding to the entry $\left[\underline{a}_{ij}, \overline{a}_{ij}\right]$ of the initial interval PM $\mathcal{A}$ (2), then the obtained fuzzy PM is denoted by $\tilde{\mathcal{A}}$ and is expressed as:

$$\tilde{\mathcal{A}} = \begin{bmatrix} \tilde{A_{11}} & \tilde{A_{12}} & \dots & \tilde{A_{1s}} \\ \tilde{A_{21}} & \tilde{A_{22}} & \dots & \tilde{A_{2s}} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{A_{r1}} & \tilde{A_{r2}} & \dots & \tilde{A_{rs}} \end{bmatrix}. \tag{13}$$

Step 2: Using ranking function 7 or 12, transform the fuzzy entries of PM into their equivalent crisp values, i.e., the fuzzy PM eventually reduces to a crisp PM given by A and is expressed as:

$$A = \begin{bmatrix} R(\tilde{A_{11}}) & R(\tilde{A_{12}}) & \dots & R(\tilde{A_{1s}}) \\ R(\tilde{A_{21}}) & R(\tilde{A_{22}}) & \dots & R(\tilde{A_{2s}}) \\ \vdots & \vdots & \ddots & \vdots \\ R(\tilde{A_{r1}}) & R(\tilde{A_{r2}}) & \dots & R(\tilde{A_{rs}}) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1s} \\ a_{21} & a_{22} & \dots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rs} \end{bmatrix}, \tag{14}$$

if the crisp equivalent of the payoff $\tilde{A_{ij}}$ of PM $\tilde{\mathcal{A}}$ is $a_{ij} = \mathfrak{R}(\tilde{A_{ij}})$.

Step 3: Look for saddle points in the crisp GP. If so, determine it along with the value of game as well as the corresponding pure strategy. Go to the next step if not.

Step 4: After Step 2, the payoffs of the game are crisp, thus the obtained crisp game can be represented as a LPP, the solution to which yields the value of game along with both the players' optimal mixed strategies. Also, it can be solved using the graphical method used for classical GPs.

Flowchart of the presented solution method is shown in Fig. 9.

4.1|Existence and uniqueness of solution by the presented method

By the process of fuzzification followed by defuzzification, we transform the IVGP into a crisp game while preserving the structure and properties of the original game. From the linear programming solutions of the equivalent crisp game, we obtain the optimal strategies.

- Existence of optimal strategies and value of game:
  By the fundamental theorem of linear programming, if the feasible region is non-empty and bounded, then every LPP has an optimal solution. Since the original IVGP is transformed to a crisp game, the optimal strategies for the players exist for the LPPs.

- Uniqueness of the solution:
  The uniqueness of the solution can be ensured if the PM A derived from the defuzzification process has properties that lead to a unique solution in the linear programming framework. Typically, uniqueness can be guaranteed if the game has a strictly determined value, i.e., the matrix A has a saddle point.

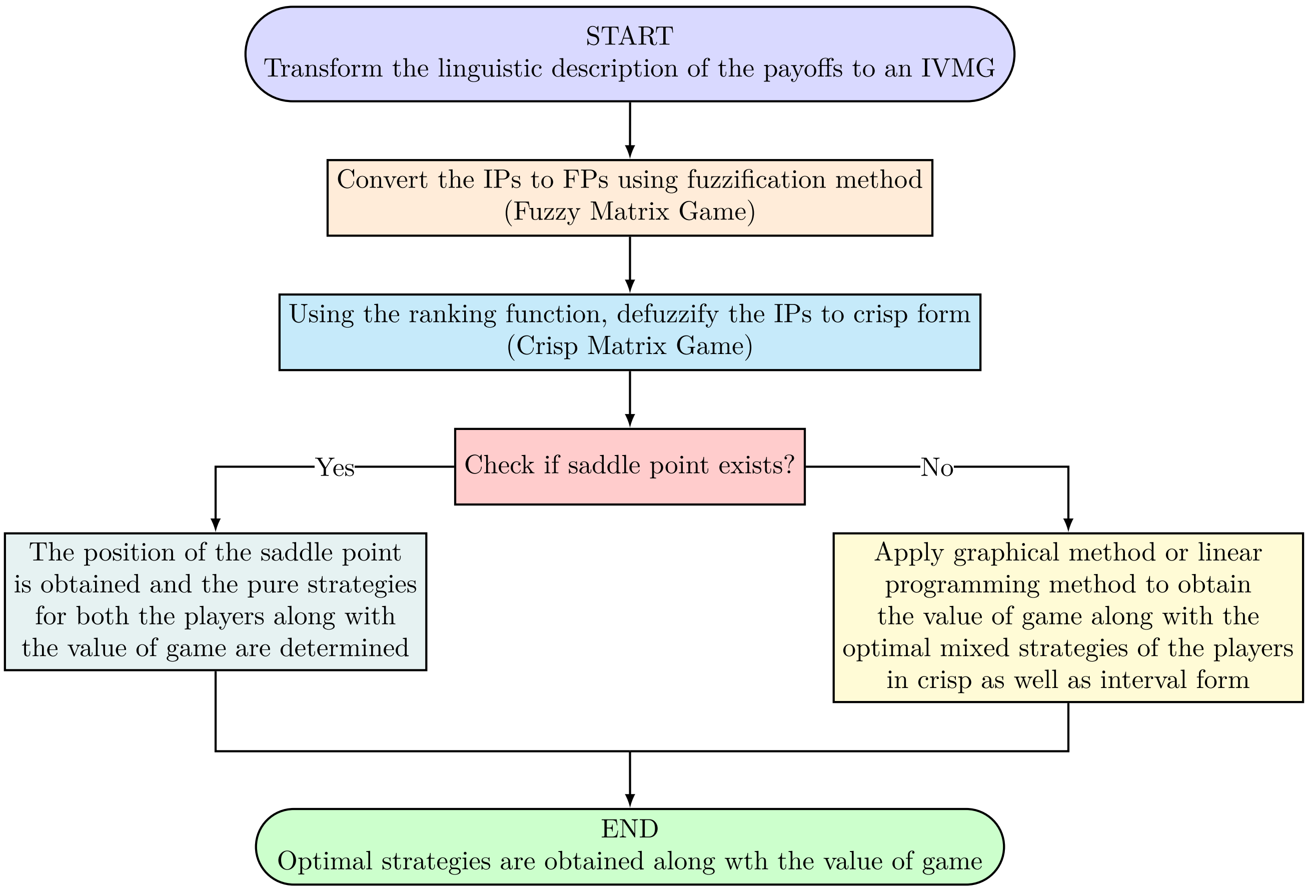

Fig. 9. Flowchart for the solution procedure

5|Numerical Examples

The use of interval values in the PM introduces an additional layer of complexity to the classic game theory problem. Traditional game theory assumes precise payoffs, but real-world scenarios often involve uncertainty. By incorporating intervals, the model provides a more realistic and practical approach to DM in competitive environments. We consider two examples for numerical illustration of the presented method for IVMG.

Note 5. *The data used in the examples is hypothetical and created for illustrative purposes to present the application of interval-valued game theory in a retail industry context. While these values do not derive from actual business data, they are designed to reflect realistic ranges of profits that competing retail chains might encounter. This hypothetical scenario allows for a clear and practical demonstration of the presented solution method for IVMG. Future research could apply this methodology to real-world data obtained from retail chains or other industries facing similar competitive and uncertain environments.*

5.1|Example 1: Real-life example in competitive retail environment

In this section, we give an example to illustrate how interval values and game theory can be combined to model and analyze strategic decisions in the retail sector, offering a robust framework for handling uncertainty and competition.

Consider a situation where two competing retail chains, Retail Chain A and Retail Chain B, are trying to maximize their profits by deciding on promotional strategies for three different product categories: Electronics, Clothing, and Groceries. The profits for each strategy are uncertain due to factors like market demand, seasonal trends, and supply chain issues, and are therefore represented as intervals. By considering a range of possible outcomes, the model captures the reality of unpredictable market conditions, allowing retail chains to plan for best-case, worst-case, and most likely scenarios. The IPs (Table 1) represent the profit in thousands of dollars for Retail Chain A and the corresponding profit for Retail Chain B. By analyzing the PM, retail chains can identify the most beneficial promotional strategies and anticipate competitive responses, optimizing their overall profit.

Table 1. PM for Example 1 with IPs.

Retail Chain B→ Retail Chain A ↓	(I) Electronics	(II) Clothing	(III) Groceries
(I) Electronics	$[2, 6]$	$[4, 7]$	$[3, 4]$
(II) Clothing	$[4, 6]$	$[5, 7]$	$[2, 6]$
(III) Groceries	$[2, 5]$	$[2, 6]$	$[3, 7]$

Each retail chain must select its promotional strategy while anticipating the potential actions of its competitor. This interplay requires strategic thinking and the ability to adapt to the competitor's decisions. For example, if Retail Chain A opts for a heavy promotion in Electronics, it must consider the possible responses from Retail Chain B in terms of its own promotional efforts across Electronics, Clothing, or Groceries. By incorporating game theory and interval analysis, retail chains can develop robust strategies to maximize their profits while accounting for the complex and uncertain nature of the market.

5.1.1|Solution:

Part 1: Using PFN

Step 1: Using the fuzzification method, the first step is to transform the interval PM into a pentagonal fuzzy PM (Table 2).

Table 2. PM for Example 1 with pentagonal FPs.

Retail Chain B→ Retail Chain A ↓	(I) Electronics	(II) Clothing	(III) Groceries
(I) Electronics	(2, 2.57, 3.71, 4.86, 6)	(4, 4.43, 5.29, 6.14, 7)	(3, 3.14, 3.43, 3.71, 4)
(II) Clothing	(4, 4.29, 4.86, 5.43, 6)	(5, 5.29, 5.86, 6.43, 7)	(2, 2.57, 3.71, 4.86, 6)
(III) Groceries	(2, 2.43, 3.29, 4.14, 5)	(2, 2.57, 3.71, 4.86, 6)	(2, 3.57, 4.71, 5.86, 7)

Using raking function given in Section 3, we transform the fuzzy pay off matrix to crisp PM (Table 3).

Table 3. Crisp PM for Example 1 using PFN.

Retail Chain B→ Retail Chain A ↓	(I) Electronics	(II) Clothing	(III) Groceries
(I) Electronics	3.8	5.3	3.5
(II) Clothing	4.9	5.9	3.8
(III) Groceries	3.4	3.8	4.8

The problem does not possess a saddle point. As all the elements of the first row are less than the corresponding elements of the second row, so the first row is dominated by the second row. Thus, on eliminating the first row, we get the resultant PM (Table 4).

Table 4. PM after applying dominance rule.

Retail Chain B→ Retail Chain A ↓	(I) Electronics	(II) Clothing	(III) Groceries
(II) Clothing	4.9	5.9	3.8
(III) Groceries	3.4	3.8	4.8

Clearly, the problem does not possess a saddle point. Let the optimum mixed strategy of Retail Chain A against Retail Chain B be $S_{\mathsf{A}} = \begin{bmatrix} \text{Clothing} & \text{Groceries} \\ p_1 & p_2 \end{bmatrix}$, where $p_1 + p_2 = 1$. Then Retail Chain A's expected pay off against Retail Chain B's pure moves are given in Table 5.

Table 5. Retail Chain A's expected pay off against Retail Chain B's pure moves

Company Q's pure move	Retail Chain A's expected payoff
(I) Electronics	$1.5\ p_1 + 3.4$
(II) Clothing	$2.1\ p_1 + 3.8$
(III) Groceries	$-1p_1 + 4.8$

We plot these expected payoffs as a function of p_1 as shown in Fig. 10.

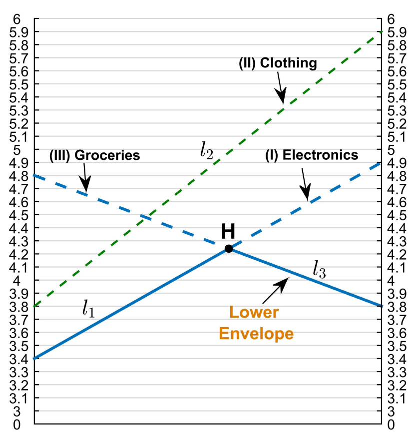

Fig. 10. Game graph

Here, H is the highest point in lower envelope of Retail Chain A's expected payoff. Lines B_2 and B_3 intersect at point H. Hence, the problem reduces to the PM given in Table 6.

Table 6. Final PM.

Retail Chain B→ Retail Chain A ↓	(I) Electronics	(III) Groceries
(II) Clothing	4.9	3.8
(III) Groceries	3.4	4.8

Let $S_{\mathsf{A}} = \begin{bmatrix} \text{Clothing} & \text{Groceries} \\ p_1 & p_2 \end{bmatrix}$ and $S_{\mathsf{B}} = \begin{bmatrix} \text{Clothing} & \text{Groceries} \\ q_1 & q_2 \end{bmatrix}$ be optimum strategies for retail chain A and retail chain B, in the same order, so that $p_1 + p_2 = 1$ and $q_1 + q_2 = 1$, then

$$p_1 = \frac{4.8 - 3.4}{4.9 + 4.8 - (3.8 + 3.4)} = \frac{1.4}{2.5} = 0.56,\ \ p_2 = 1 - p_1 = 1 - 0.56 = 0.44,$$

$$\text{and } q_1 = \frac{4.8 - 3.8}{4.9 + 4.8 - (3.8 + 3.4)} = \frac{1}{2.5} = 0.4,\ \ q_2 = 1 - q_1 = 1 - 0.4 = 0.6.$$

Hence, the solution is

$$S_{\mathsf{A}} = \begin{bmatrix} \text{Electronics} & \text{Clothing} & \text{Groceries} \\ 0 & 0.56 & 0.44 \end{bmatrix} \text{ and } S_{\mathsf{B}} = \begin{bmatrix} \text{Electronics} & \text{Clothing} & \text{Groceries} \\ 0.4 & 0 & 0.6 \end{bmatrix}$$

and the expected value of game is 4.24. In interval form, the value of game is $[2.712, 6.088]$.

Part 2: Using Hexagonal Fuzzy Number

By applying the fuzzification presented for HFN, the GP reduces to the PM given by Table 7.

Table 7. PM for Example 1 with hexagonal FPs.

Retail Chain B→ Retail Chain A ↓	(I) Electronics	(II) Clothing	(III) Groceries
(I) Electronics	(2, 2.44, 3.33, 4.22, 5.11, 6)	(4, 4.33, 5.00, 5.67, 6.33, 7)	(3, 3.11, 3.33, 3.56, 3.78 , 4)
(II) Clothing	(4, 4.22, 4.67, 5.11, 5.56, 6)	(5, 5.22,5.67, 6.11, 6.56, 7)	(2, 2.44, 3.33, 4.22, 5.11, 6)
(III) Groceries	(2, 2.33, 3.00, 3.67, 4.33, 5)	(2, 2.44, 3.33, 4.22, 5.11, 6)	(2, 3.44, 4.33, 5.22, 6.11, 7)

By applying the ranking function, the given problem is reduced to a crisp form, as shown in Table 8.

Table 8. Crisp PM for Example 1 using HFN.

Retail Chain B→ Retail Chain A ↓	(I) Electronics	(II) Clothing	(III) Groceries
(I) Electronics	3.8	5.4	3.5
(II) Clothing	4.9	5.9	3.8
(III) Groceries	3.4	3.8	4.8

By applying the method used for PFN, the problem has exactly same solution. Thus, Retail Chain A should allocate its promotional budget approximately as follows:

- 0% to Electronics
- 56% to Clothing
- 44% to Groceries

Retail Chain B should allocate its promotional budget approximately as follows:

- 40% to Electronics
- 0% to Clothing
- 60% to Groceries

These derived strategies ensure that both Retail Chain A and Retail Chain B maximize their respective profits under uncertainty. Also, the value of the game, representing the expected profit for both chains given their optimal strategies, is 4240 dollars. In interval form, the value of game is between 2712 dollars and 6088 dollars.

5.2|Example 2

Consider two person zero-sum game given in Table 9.

Table 9. PM for Example 2 with IPs.

Company Q→ Company P ↓	(I) No ad	(II) Medium ad	(III) Heavy ad
(I) No ad	$[6, 8]$	$[1, 5]$	$[3, 8]$
(II) Medium ad	$[4, 7]$	$[9, 12]$	$[6, 10]$
(III) Heavy ad	$[5, 8]$	$[2, 5]$	$[8, 13]$

5.2.1|Solution:

Part 1-Using PFN

By applying the fuzzification method for PFN, the problem reduces to Table 10.

Table 10. PM for Example 2 with pentagonal FPs.

Company Q→ Company P ↓	(I) No ad	(II) Medium ad	(III) Heavy ad
(I) No ad	(6, 6.29, 6.86, 7.43, 8)	(1, 1.57, 2.71, 3.86, 5)	(3, 3.71, 5.14, 6.57, 8)
(II) Medium ad	(4, 4.43, 5.29, 6.14, 7)	(9, 9.43, 10.29, 11.14, 12)	(6, 6.57, 7.71, 8.86, 10)
(III) Heavy ad	(5, 5.43, 6.29, 7.14, 8)	(2, 2.43, 3.29, 4.14, 5)	(8, 8.71, 10.14, 11.57, 13)

The crisp form of the above fuzzy problem, which is obtained by utilizing the ranking function given in Section 3.2, is given by Table 11.

Table 11. Crisp PM for Example 2 using PFN.

Company Q→ Company P ↓	(I) No ad	(II) Medium ad	(III) Heavy ad
(I) No ad	6.9	2.8	5.2
(II) Medium ad	5.3	10.3	7.8
(III) Heavy ad	6.3	3.4	10.2

Let the strategies of two players be: $S_{\mathsf{P}} = \begin{bmatrix} P_1 & P_2 & P_3 \\ p_1 & p_2 & p_3 \end{bmatrix}$ and $S_{\mathsf{Q}} = \begin{bmatrix} Q_1 & Q_2 & Q_3 \\ q_1 & q_2 & q_3 \end{bmatrix}$, where $p_1 + p_2 + p_3 = 1$ and $q_1 + q_2 + q_3 = 1$. For the above problem, the LPP formulation is:
Let 'u' be expected minimum gain to company P and 'v' be expected minimum loss to company Q. Also, let $x_{\mathsf{i}} = p_{\mathsf{i}}/u$ and $y_{\mathsf{i}} = q_{\mathsf{i}}/v$, then
For company P:

$$\text{Maximize } u \equiv \text{Minimize } \frac{1}{u} = x_1 + x_2 + x + 3$$

subject to

$$6.9\ x_1 + 5.3\ x_2 + 6.3x_3 \geq 1;$$
$$2.8\ x_1 + 10.3\ x_2 + 3.4\ x_3 \geq 1;$$
$$5.2\ x_1 + 7.8\ x_2 + 10.2\ x_3 \geq 1.$$

For company Q:

$$\text{Minimize } v\ \equiv\ \text{Maximixe } \frac{1}{v}\ =\ y_1\ +\ y_2\ +\ y_3$$

subject to

$$6.9y_1 + 2.8y_2 + 5.2y_3 \leq 1;$$
$$5.3y_1 + 10.3y_2 + 7.8y_3 \leq 1;$$
$$6.3y_1 + 3.4y_2 + 10.2y_3 \leq 1.$$

By applying the dual simplex method, the final table obtained is given by Table 12.

Table 12. Solution using simplex method.

Final Iteration		c_j	1	1	1	0	0	0
B	$\mathbf{C_B}$	$\mathbf{X_B}$	x_1	x_2	x_3	s_1	s_2	s_3
s_3	0	0.0311	0	0	-5.0583	-0.5641	-0.467	1
x_2	1	0.0729	0	1	0.1035	0.0498	-0.1227	0
x_1	1	0.0889	1	0	0.8335	-0.1832	0.0943	0
$\mathbf{Z = 0.1618}$		Z_j	**1**	**1**	**0.937**	**-0.1334**	**-0.0285**	**0**
		$Z_j - C_j$	0	0	0.063	-0.1334	-0.0285	0

Hence, we have $x_1 = 0.0889$, $x_2 = 0.0729$, $x_3 = 0$ & $u = 0.1618$.

Using duality, $y_1 = 0.1334$, $x_2 = 0.0285$, $x_3 = 0$ & $v = 0.1618$. Thus,

$$S_P = \begin{bmatrix} P_1 & P_2 & P_3 \\ \frac{889}{1618} & \frac{729}{1618} & 0 \end{bmatrix} \text{ and } S_Q = \begin{bmatrix} Q_1 & Q_2 & Q_3 \\ \frac{1334}{1618} & \frac{285}{1618} & 0 \end{bmatrix} \text{ and } \mathbf{v = 6.177}.$$

Part 2: Using HFNs

By applying the fuzzification method for HFNs, the problem reduces to PM given by Table 13,

Table 13. PM for Example 2 with hexagonal FPs.

Company Q→ Company P ↓	(I) No ad	(II) Medium ad	(III) Heavy ad
(I) No ad	(6, 6.22, 6.67, 7.11, 7.56, 8)	(1, 1.44, 2.33, 3.22, 4.11, 5)	(3, 3.56, 4.67, 5.78, 6.89, 8)
(II) Medium ad	(4, 4.33, 5.00, 5.67, 6.33, 7)	(9, 9.33, 10.00, 10.67, 11.33, 12)	(6, 6.44, 7.33, 8.22, 9.11, 10)
(III) Heavy ad	(5, 5.33, 6.00, 6.67, 7.33, 8)	(2, 2.33, 3.00, 3.67, 4.33, 5)	(8, 8.56, 9.67, 10.78, 11.89, 13)

By applying the ranking function, the crisp valued PM corresponding to the above fuzzy GP is given by Table 14.

Table 14. Crisp PM for Example 2 using HFN.

Company Q→ Company P ↓	(I) No ad	(II) Medium ad	(III) Heavy ad
(I) No ad	6.9	2.8	5.3
(II) Medium ad	5.3	10.4	7.8
(III) Heavy ad	6.4	3.4	10.2

Following exactly the same steps as above, the solution is obtained as:

$$S_P = \begin{bmatrix} P_1 & P_2 & P_3 \\ \frac{883}{1607} & \frac{724}{1607} & 0 \end{bmatrix} \text{ and } S_Q = \begin{bmatrix} Q_1 & Q_2 & Q_3 \\ \frac{1342}{1607} & \frac{265}{1607} & 0 \end{bmatrix} \text{ and } v = 6.22.$$

6|Computational results obtained by other existing methods in the literature for Example 1 and its comparative analysis

1. By applying the linear programming method presented by Li [28], LPMs obtained for Retail Chain A are:

$$\begin{aligned} &\text{Min } (\overline{x_1}+\overline{x_2}+\overline{x_3})\\ \text{subject to } &6\overline{x_1}+6\overline{x_2}+5\overline{x_3}\geq 1;\\ &7\overline{x_1}+7\overline{x_2}+6\overline{x_3}\geq 1;\\ &4\overline{x_1}+6\overline{x_2}+7\overline{x_3}\geq 1;\\ &\overline{x_i}\geq 0\ (i=1,2,3). \end{aligned} \qquad \text{(LPP1)}$$

$$\begin{aligned} &\text{Min } (\underline{x_1}+\underline{x_2}+\underline{x_3})\\ \text{subject to } &2\underline{x_1}+4\underline{x_2}+2\underline{x_3}\geq 1;\\ &4\underline{x_1}+5\underline{x_2}+2\underline{x_3}\geq 1;\\ &3\underline{x_1}+2\underline{x_2}+3\underline{x_3}\geq 1;\\ &\underline{x_i}\geq 0\ (i=1,2,3). \end{aligned} \qquad \text{(LPP2)}$$

By applying the simplex method for LPP1 and LPP2, the optimal solutions obtained are

$$\overline{x_1}^*=0,\ \overline{x_2}^*=0.1667,\ \overline{x_3}^*=0\ \&\ \underline{x_1}^*=0,\ \underline{x_2}^*=0.125,\ \underline{x_3}^*=0.25.$$

Using these values, the optimal strategies are obtained as:

$$\overline{s_1}^*=0,\ \overline{s_2}^*=1,\ \overline{s_3}^*=0\ \&\ \underline{s_1}^*=0,\ \underline{s_2}^*=0.333,\ \underline{s_3}^*=0.667.$$

and the value of game for Retail Chain A, can be presented as closed interval by $[\underline{\nu},\overline{\nu}]=[2.667,5.998]$.

For Retail Chain B, the LPMs are:

$$\begin{aligned} &\text{Max } (\overline{y_1}+\overline{y_2}+\overline{y_3})\\ \text{subject to } &6\overline{y_1}+7\overline{y_2}+4\overline{y_3}\leq 1;\\ &6\overline{y_1}+7\overline{y_2}+6\overline{y_3}\leq 1;\\ &5\overline{y_1}+6\overline{y_2}+7\overline{y_3}\leq 1;\\ &\overline{y_i}\geq 0\ (i=1,2,3). \end{aligned} \qquad \text{(LPP3)}$$

$$\begin{aligned} &\text{Max } (\underline{y_1}+\underline{y_2}+\underline{y_3})\\ \text{subject to } &2\underline{y_1}+4\underline{y_2}+53\underline{y_3}\leq 1;\\ &4\underline{y_1}+5\underline{y_2}+2\underline{y_3}\leq 1;\\ &2\underline{y_1}+2\underline{y_2}+3\underline{y_3}\leq 1;\\ &\underline{y_i}\geq 0\ (i=1,2,3). \end{aligned} \qquad \text{(LPP4)}$$

By applying the simplex method for LPP3 and LPP4, the optimal solutions obtained are:

$$\overline{y_1}^*=0.1667,\ \overline{y_2}^*=0,\ \overline{y_1}^*=0\ \&\ \underline{y_1}^*=0.125,\ \underline{y_2}^*=0,\ \underline{y_3}^*=0.25.$$

Using these values, the optimal strategies are obtained as:

$$\overline{t_1}^*=1,\ \overline{t_2}^*=0,\ \overline{t_3}^*=0\ \&\ \underline{t_1}^*=0.333,\ \underline{t_2}^*=0,\ \underline{t_3}^*=0.667.$$

and the value of game obtained for Retail Chain B is $\left[\underline{\mu},\overline{\mu}\right]=[2.667,5.998]$.
Since, $[\underline{\nu},\overline{\nu}]=\left[\underline{\mu},\overline{\mu}\right]=[2.667,5.998]$, hence the value of game is $\left[\underline{V},\overline{V}\right]=[2.667,5.998]$.
Here, the game value is only obtained in the form of an interval, i.e., it also possess uncertainty. However, the approach presented in this work gives crisp value of the game also.

2. By applying the the method presented by Liu and Kao [32], the LPP obtained for Retail Chain A is given by equations (1) and (2). Here, the value of game is given by $\left[\underline{\nu}^{\mathsf{LK}},\overline{\nu}^{\mathsf{LK}}\right]$, whose lower bound is expressed as $\underline{\nu}^{\mathsf{LK}}=Min\ (\underline{\mu}^{\mathsf{LK}})$ and upper bound is expressed as $\overline{\nu}^{\mathsf{LK}}=Max\ (\overline{\mu}^{\mathsf{LK}})$.

$$
\begin{array}{ll}
\text{Max } (\overline{\mu}^{\mathsf{LK}}) & \\
\text{subject to} & \\
p_{11}+p_{21}+p_{31} \geq \overline{\mu}^{\mathsf{LK}}; & \\
p_{12}+p_{22}+p_{32} \geq \overline{\mu}^{\mathsf{LK}}; & \\
p_{13}+p_{23}+p_{33} \geq \overline{\mu}^{\mathsf{LK}}; & \\
2y_1 \leq p_{11} \leq 6y_1; & \\
4y_1 \leq p_{12} \leq 7y_1; & \\
3y_1 \leq p_{13} \leq 4y_1; & \\
4y_2 \leq p_{21} \leq 6y_2; & \\
5y_2 \leq p_{22} \leq 7y_2; & (1) \\
2y_2 \leq p_{23} \leq 6y_2; & \\
4y_3 \leq p_{31} \leq 5y_3; & \\
2y_3 \leq p_{32} \leq 6y_3; & \\
3y_3 \leq p_{33} \leq 7y_3; & \\
y_1+y_2+y_3 = 1; & \\
y_{\mathsf{i}} \geq 0 \ (i=1,2,3) & \\
\overline{\mu}^{\mathsf{LK}} \ \ and \ p_{\mathsf{ij}} \text{ are unrestricted in sign} & \\
(i=1,2,3; \ j=1,2,3). &
\end{array}
$$

$$
\begin{array}{ll}
\text{Min } (\underline{\mu}^{\mathsf{LK}}) & \\
\text{subject to} & \\
q_{11}+q_{12}+q_{13} \leq \underline{\mu}^{\mathsf{LK}}; & \\
q_{21}+q_{22}+q_{23} \leq \underline{\mu}^{\mathsf{LK}}; & \\
q_{31}+q_{32}+q_{33} \leq \underline{\mu}^{\mathsf{LK}}; & \\
2z_1 \leq q_{11} \leq 6z_1; & \\
4z_1 \leq q_{21} \leq 6z_1; & \\
2z_1 \leq q_{31} \leq 5z_1; & \\
4z_2 \leq q_{12} \leq 7z_2; & \\
5z_2 \leq q_{22} \leq 7z_2; & (2) \\
2z_2 \leq q_{32} \leq 6z_2; & \\
3z_3 \leq q_{13} \leq 4z_3; & \\
2z_3 \leq q_{23} \leq 6z_3; & \\
3z_3 \leq q_{33} \leq 7z_3; & \\
z_1+z_2+z_3 = 1; & \\
z_{\mathsf{i}} \geq 0; \ (i=1,2,3) & \\
\underline{\mu}^{\mathsf{LK}} \ \ and \ q_{\mathsf{ij}} \text{ are unrestricted in sign} & \\
(i=1,2,3; \ j=1,2,3). &
\end{array}
$$

The optimal solution obtained on solving equation (1) by LINGO, is

$$Max\ (\overline{\mu}^{\mathsf{LK}}) = 6,\ y_1{}^* = 0,\ y_2{}^* = 1,\ y_3{}^* = 0$$

and for equation (2) the optimal solution obtained is

$$Min\ (\underline{\mu}^{\mathsf{LK}}) = 2.667,\ z_1{}^* = 0.333,\ z_2{}^* = 0.667,\ z_3{}^* = 0.$$

Hence, value of game for Retail Chain A is $[2.667, 6]$.

Similar LPPs are derived for Retail Chain B, yielding a game value of $[2.667, 6]$, identical to that obtained for Retail Chain A. Therefore, the game value is $[2.667, 6]$.

3. Shashikhin [46] used interval arithmetic defined by Moore [35] to define generalized maximum and minimum operators of the intervals. According to this method, for Example 1, we have

$\min\limits_{1 \leq j \leq 3} \left\{ \left[\underline{a}_{1\mathsf{j}}, \overline{a}_{1\mathsf{j}}\right] \right\} = min\left\{[2,6], [4,7], [3,4]\right\} = [2,4]$,

$\min\limits_{1 \leq j \leq 3} \left\{ \left[\underline{a}_{2\mathsf{j}}, \overline{a}_{2\mathsf{j}}\right] \right\} = min\left\{[4,6], [5,7], [2,6]\right\} = [2,6]$,

$\min\limits_{1 \leq j \leq 3} \left\{ \left[\underline{a}_{3\mathsf{j}}, \overline{a}_{3\mathsf{j}}\right] \right\} = min\left\{[2,5], [2,6], [3,7]\right\} = [2,5]$.

Hence, $\max\limits_{1 \leq i \leq 3} \min\limits_{1 \leq j \leq 3} \left\{ \left[\underline{a}_{\mathsf{ij}}, \overline{a}_{\mathsf{ij}}\right] \right\} = [2,6]$. Similarly, $\min\limits_{1 \leq i \leq 3} \max\limits_{1 \leq j \leq 3} \left\{ \left[\underline{a}_{\mathsf{ij}}, \overline{a}_{\mathsf{ij}}\right] \right\} = [2,6]$.

Since, $\max\limits_{1 \leq i \leq 3} \min\limits_{1 \leq j \leq 3} \left\{ \left[\underline{a}_{\mathsf{ij}}, \overline{a}_{\mathsf{ij}}\right] \right\} = \min\limits_{1 \leq i \leq 3} \max\limits_{1 \leq j \leq 3} \left\{ \left[\underline{a}_{\mathsf{ij}}, \overline{a}_{\mathsf{ij}}\right] \right\} = [2,6]$. Thus, the value of game is $\left[\underline{V}^{\mathsf{S}}, \overline{V}^{\mathsf{S}}\right] = [2,6]$.

However, this method does not always guarantee that the IVMG will have a value. For instance, in Example 2,

$$\max_{1 \leq i \leq 3} \min_{1 \leq j \leq 3} = [4,7] \text{ and } \min_{1 \leq i \leq 3} \max_{1 \leq j \leq 3} = [6,8].$$

That is, $\max\limits_{1 \leq i \leq 3} \min\limits_{1 \leq j \leq 3} \neq \min\limits_{1 \leq i \leq 3} \max\limits_{1 \leq j \leq 3}$. Thus, we cannot determine the game value here.

However, the method presented in this work always ensures that the game value for IVMG exists.

4. By applying the method based on weighted average, as given by Li et al. [31], the upper as well as the lower bound of the game value for Retail Chain A can be obtained by solving the LPP (3).
   On taking the value of parameter α as 0.7 and solving (3) using LINGO, the optimal solution obtained is $\underline{\nu}^* = 2.667,\ \overline{\nu}^* = 3.952,\ y_1^* = 0.33..,\ y_2^* = 0.33..,\ y_3^* = 0.33$. Thus, the gain floor for Retail Chain A is $\tilde{\nu} = [\underline{\nu}^*, \overline{\nu}^*] = [2.667, 3.952]$.

   Similarly, the upper and lower constraints of the loss ceiling and the related optimal strategy of Retail Chain B is obtained by finding the optimal solution of (4).
   On solving (4) and taking $\alpha = 0.7$, the optimal strategy $z_1^* = 0.435,\ z_2^* = 0,\ z_3^* = 0.565$ and upper and lower constraints of the loss ceiling $\underline{\mu}^* = 2.5652,\ \overline{\mu}^* = 6.1304$ are obtained for Retail Chain B.
   Hence, loss ceiling for Q is $\tilde{\omega} = [\underline{\omega}^*, \overline{\omega}^*] = [2.5652, 6.1304]$.

$$
\begin{aligned}
&Max\ (3\underline{\nu} + \overline{\nu})/4 \\
\text{subject to}\quad & \\
&2y_1 + 4y_2 + 2y_3 \geq \underline{\nu}; \\
&4y_1 + 5y_2 + 2y_3 \geq \underline{\nu}; \\
&3y_1 + 2y_2 + 3y_3 \geq \underline{\nu}; \\
&(1-\alpha)(6y_1 + 6y_2 + 5y_3) + \alpha(2y_1 + 4y_2 + 2y_3) \geq (1-\alpha)\underline{\nu} + \alpha\overline{\nu}; \\
&(1-\alpha)7y_1 + 7y_2 + 6y_3 + \alpha(4y_1 + 5y_2 + 2y_3) \geq (1-\alpha)\underline{\nu} + \alpha\overline{\nu}; \\
&(1-\alpha)4y_1 + 6y_2 + 7y_3 + \alpha(3y_1 + 2y_2 + 3y_3) \geq (1-\alpha)\underline{\nu} + \alpha\overline{\nu}; \\
&\underline{\nu} \leq \overline{\nu}; \\
&y_1 + y_2 + y_3 = 1; \\
&y_{\mathrm{i}} \geq 0\ (i = 1, 2, 3); \\
&\overline{\nu}\ and\ \underline{\nu}\ \text{are unrestricted in sign.}
\end{aligned}
\tag{3}
$$

On solving the following given LPP (4), the game value for Retail Chain B can be easily obtained:

$$
\begin{aligned}
&Min\ (3\underline{\omega} + \overline{\omega})/4 \\
\text{subject to}\quad & \\
&6z_1 + 7z_2 + 4z_3 \leq \overline{\omega}; \\
&6z_1 + 7z_2 + 6z_3 \leq \overline{\omega}; \\
&5z_1 + 6z_2 + 7z_3 \leq \overline{\omega}; \\
&(1-\alpha)(2z_1 + 4z_2 + 3z_3) + \alpha(6z_1 + 7z_2 + 4z_3) \leq (1-\alpha)\underline{\omega} + \alpha\overline{\omega}; \\
&(1-\alpha)(4z_1 + 5z_2 + 2z_3) + \alpha(6z_1 + 7z_2 + 6z_3) \leq (1-\alpha)\underline{\omega} + \alpha\overline{\omega}; \\
&(1-\alpha)(2z_1 + 2z_2 + 3z_3) + \alpha(5z_1 + 6z_2 + 7z_3) \leq (1-\alpha)\underline{\omega} + \alpha\overline{\omega}; \\
&\underline{\omega} \leq \overline{\omega}; \\
&z_1 + z_2 + z_3 = 1; \\
&z_{\mathrm{i}} \geq 0;\ (i = 1, 2, 3) \\
&\underline{\omega}\ and\ \overline{\omega}\ \text{are unrestricted in sign.}
\end{aligned}
\tag{4}
$$

Based on the interval comparison's acceptability index provided by Li et al. [28] in Definition 3, it follows that,

$$\phi(\tilde{\nu} \leq_{\mathrm{t}} \tilde{\omega}) = (\overline{\omega} - \overline{\nu}) / \left[2(r(\tilde{\omega}) - r(\tilde{\nu})\right];\ r(\tilde{\omega}) = (\overline{\omega} - \underline{\omega})/2\ and\ r(\tilde{\nu}) = (\overline{\nu} - \underline{\nu})/2.$$

Thus, $\phi(\tilde{\nu} \leq_{\mathrm{t}} \tilde{\omega}) = 0.95$, i.e., the statement "Retail Chain A's gain floor is not greater than Retail Chain B's loss ceiling" is valid and it has an acceptability degree of 0.95.

7|Conclusion

This paper presents a study on two person zero-sum GP in which the entries of the PM are considered in the form of intervals. We presented a simple and efficient technique to obtain the optimal strategies for each players and the game's value. In this presented method, we first fuzzify the payoffs (originally in interval form) into pentagonal or HFNs, depending on the information given by the decision maker. Then using the ranking function for the aforementioned FNs, we defuzzify the FPs to crisp form. In this way, the interval GP is transformed into an equivalent crisp game. The solution of this crisp GP is then obtained by leveraging the equivalence between crisp rectangular games and LPPs. The pair of LPPs is solved using simplex method or LINGO. The comparison of the game value obtained through our presented method is then done with solution obtained by the methods presented in the past. The comparison suggests that the present approach is easier and numerically efficient than some methods. Also, unlike some existing methods, it ensures that IVMG has a value.

Although, the game value obtained through the method presented in this paper can be expressed both in interval form and crisp form; however, there is a drawback of this approach, which is, that the optimal strategies can only be derived in crisp form. In contrast, some existing approaches in the literature offer both the game value and the optimal strategies in the same format as the input data (interval form). In spite of that, these previously introduced methods also have significant limitations, which are briefly discussed in Section 6. Thus, future research could focus on modifying the presented algorithm to yield the optimal strategies for IVMGs in interval form. Additionally, the fuzzification and defuzzification approach presented could be applied to inventory modeling, asymmetric games, cooperative GPs, and infinite games. Furthermore, creating codes for the existing, present and new algorithms for GPs in various uncertain environments could be a promising area for research. This advancement would facilitate a direct comparison of the effectiveness of numerous algorithms regarding time and computational complexity.

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Author Contribution

M. Bisht: methodology, software, and editing. J. Ahmad: conceptualization. S. Rawat: writing and editing. All authors have read and agreed to the published version of the manuscript.

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Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings. Funders played no role in the study's design, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

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